XVI Konferencja Naukowo-Techniczna
TKI2022
TECHNIKI KOMPUTEROWE W INŻYNIERII
18–21 października 2022

A modified kernel method for solving engineering problems with interfaces

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ABSTRACT: The paper deals with a modification of a kind of meshless method to properly solve differential equations describing some engineering problems with discontinuities at interfaces. In the method the sought solution is approximated by global interpolant composed of kernel functions and with the use of the collocation procedure the differential equation is discretized. Such an approach is well suited for equations that possess smooth solutions. To extend the usefulness of the method onto engineering problems with interfaces, in the present paper another type of the approximation of the sought solution is applied. The interpolation with so-called variably scaled discontinuous kernels is introduced into the method. The paper shows that this approach with properly applied interface conditions gives very accurate results, what is presented by some numerical tests.

KEYWORDS: meshless method, kernel function, interface problem, discontinuities

1. Introduction

In recent years an importance of meshless discretization techniques has significantly raised [1, 2]. Their main advantage over well-known mesh techniques is the possibility of the use of scattered nodes to discretize the domain. Among these meshless techniques there is a wide group of methods that takes advantage of kernel functions to construct an approximation of the solution [3]. Owing to these functions the treatment of the problems defined in higher dimensions is straightforward, almost the same as in one dimension. The most popular kernels are radial basis functions that have been recently applied to solve many problems in science and engineering [4]. It was found that using global interpolation with such basis functions one can expect rapid convergence, even exponential one under certain conditions [5]. But on the other hand this type of approximation is well suited for problems possessing smooth solutions. In engineering, there are a lot of problems possessing local features that are characterized by non-smooth solutions. For example a heat transfer in a plate composed of two or more different materials separated by interfaces.

Therefore in the present work an effort is made to extend the usefulness of a kernel method onto the problems possessing discontinuities on the interfaces. To this end a typical approximation of the sought solution with kernel functions is replaced by so-called variably scaled discontinuous kernels (VSDKs). The latter are recently developed functions [6] that are used so far in approximation problems [7]. In the present work the VSDK are adapted to a kernel method for solving differential equations with interfaces.

2. Kernel functions in solving differential equations

Kernel function is understood as a real-valued function of two variables coming from d-dimensional space, i.e. $K:\Omega\times\Omega\to\mathbb{R},\Omega\in\mathbb{R}^d$. With the use of this function one can conveniently construct data-dependent basis, which is a crucial point in scattered data approximation problems defined in higher dimensions. On a given set of nodes $\mathbf{x}_i$, i=1,...,N, $\mathbf{x}_i\subset\Omega\in\mathbb{R}^d$ the interpolant, which can also be viewed as a sought solution for a differential equation, has the form

$$u_h(\mathbf{x})=\sum_{j=1}^{N}c_jK(\mathbf{x},\mathbf{x}_j)=\mathbf{k}(\mathbf{x})^T\mathbf{c} \tag{1}$$

where $\mathbf{k}(\mathbf{x})$ and $\mathbf{c}$ are vectors of base functions and interpolation coefficients, respectively.

In the modified kernel method a typical kernel function is replaced by the VSDK. The latter is constructed by augmenting the dimensionality of the arguments of the original kernel with a scale function $\psi:\Omega\to\mathbb{R}$, what can be written as

$$K_\psi(\mathbf{x},\mathbf{x}_j)=K((\mathbf{x},\psi(\mathbf{x})),(\mathbf{x}_j,\psi(\mathbf{x}_j))) \tag{2}$$

To accurately reflect the discontinuities, the scale function is assumed to be a piecewise constant function with discontinuities at interfaces of a given problem.

Using the definition of the VSDK, Eq. (1) is introduced to a given interface boundary-value problem, which can be put in a general form as

$$\begin{aligned}&Lu=f\ \text{ in }\Omega,\quad Bu=g\ \text{ on }\partial\Omega,\\&[u]=w,\ [Iu]=v\ \ on\ \Gamma\end{aligned} \tag{3}$$

where the second line defines physically justified interface conditions. With the collocation procedure Eq. (3) is reduced to a system of algebraic equations, which is solved for interpolation coefficients **c**. Finally, we have algebraic approximate solution in the form of Eq. (1) that accurately reflects discontinuities.

3. Numerical examples

To show the usefulness of the method several interface problems have been solved. In Fig. 1 the solution of one-dimensional convection-diffusion problem with two interfaces is shown, while in Fig. 2 its derivative is presented. The discontinuity is due to a piecewise constant diffusion coefficient, what leads to so-called weak discontinuity – non-smooth continuous solution and discontinuous derivative.

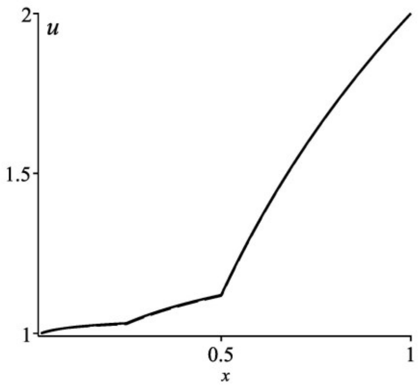

Fig. 1. Solution of one-dimensional interface problem: approximate solution – solid line, exact one- dash line.

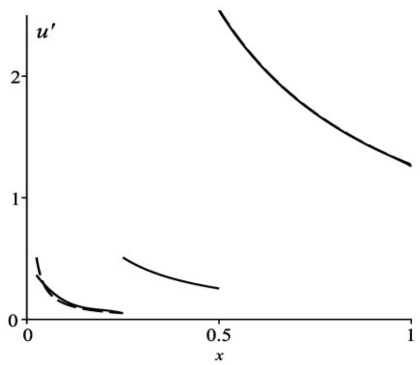

Fig. 2. Derivative of one-dimensional interface problem: approximate one – solid line, exact one- dash line.

In Fig. 3 the solution of a diffusion problem defined on two-dimensional domain is presented. The interface that divides the whole domain into two subdomains with different values of diffusion coefficient is defined on a circle centered at the origin of the coordinate system. The error of the solution is presented in Fig. 4.

The figures show that the method enables to obtain very accurate results and catch sharply local features.

4. Conclusion

The kernel collocation method has many advantages: rapid convergence, easy use in high dimensional problems and simple implementation but is not able to accurately

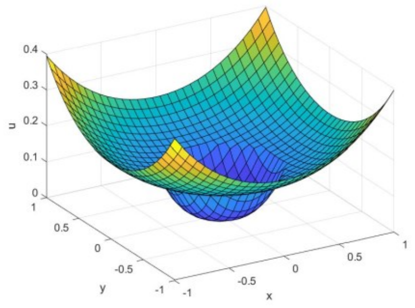

Fig. 3. Approximate solution of two-dimensional interface problem.

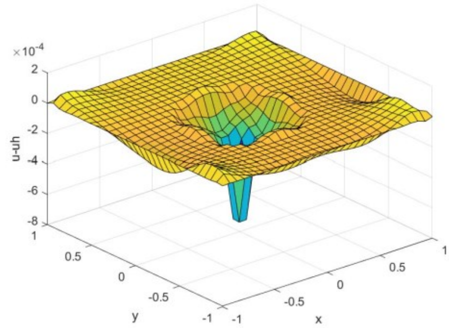

Fig. 4. Error of the solution of two-dimensional interface problem.

catch some local features. In the present work the method is extended to be useful in interface problems.

The main idea is to extend the dimensionality of the arguments of the kernel function with the scale function. By appropriate switching the value of the scale function, dependently on the position of the node, we can accurately reflect the discontinuities. The general procedure of the method is not changed. The only change is the introduction of other base functions that, owing to the proper assumption of the scale function, contain the information about local features.

Using this method a special attention should be paid to controlling the conditioning of the algebraic equation system. To this end, in the present work, an appropriate technique has been applied to find a proper value of a parameter of the method, what allows accurately to solve these equations.

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